ON THE HESSIAN OF A PRODUCT OF LINEAR FUNCTIONS.

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In a recent number of the Annals (Vol. V, p. 17) Mr. James McMahon points out that the Hessian of the binary quantic

$$a_0\,(x - a_1 y)\,(x - a_2 y)\,\ldots\,(x - a_n y)$$

is equal (apart from a numerical factor) to

$$a_0{}^2\,\Sigma\,(a_1 - a_2)^2\,(x - a_3 y)^2\,(x - a_4 y)^2\,\ldots\,(x - a_n y)^2\,;$$

this equality appearing at once from the consideration that the foregoing expression is a covariant of the same degorder as the Hessian, and that there exists but one covariant of that degorder.

The equality is not at all obvious on a direct examination or computation of the Hessian. On the other hand, it may be directly deduced by a method which applies with equal facility to a product of homogeneous linear functions in any number of variables, and which therefore gives an extension of the above theorem.

Let

$$u = L_1 L_2 \ldots L_n,$$

where $L_1, L_2, \ldots$ are linear quantics in k variables; say

$$L_1 = a_1 x + b_1 y + c_1 z + \ldots, \quad L_2 = a_2 x + b_2 y + c_2 z + \ldots, \quad \text{etc.}$$

Then

$$\frac{du}{dx} = u\left[\frac{a_1}{L_1} + \frac{a_2}{L_2} + \ldots + \frac{a_n}{L_n}\right],$$

$$\frac{d^2u}{dx^2} = u\left\{\left[\frac{a_1}{L_1} + \frac{a_2}{L_2} + \ldots + \frac{a_n}{L_n}\right]^2 - \left[\left[\frac{a_1}{L_1}\right]^2 + \left[\frac{a_2}{L_2}\right]^2 + \ldots + \left[\frac{a_n}{L_n}\right]^2\right]\right\},$$

$$\frac{d^2u}{dxdy} = u\left[\left[\frac{a_1}{L_1} + \frac{a_2}{L_2} + \ldots + \frac{a_n}{L_n}\right]\left[\frac{b_1}{L_1} + \frac{b_2}{L_2} + \ldots + \frac{b_n}{L_n}\right]\right.$$
$$\left. - \left[\frac{a_1}{L_1}\cdot\frac{b_1}{L_1} + \frac{a_2}{L_2}\cdot\frac{b_2}{L_2} + \ldots + \frac{a_n}{L_n}\cdot\frac{b_n}{L_n}\right]\right];$$

or, writing $\frac{a_i}{L_i} = \alpha_i$, $\frac{b_i}{L_i} = \beta_i$, etc.,

$$\frac{d^2u}{dx^2} = u\,[(\Sigma\alpha)^2 - \Sigma\alpha^2], \qquad \frac{d^2u}{dxdy} = u\,[\Sigma\alpha\,.\,\Sigma\beta - \Sigma\alpha\beta]\,;$$

whence it is plain that, writing H for the determinant of second derivatives (and not for that determinant divided by a numerical factor),

$$\frac{H}{(-u)^k} = \left|\begin{array}{cccccc} \Sigma\alpha & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \Sigma\beta & \beta_1 & \beta_2 & \ldots & \beta_n \\ \Sigma\gamma & \gamma_1 & \gamma_2 & \ldots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \multicolumn{5}{c}{k \text{ rows}} \end{array}\right| \cdot \left|\begin{array}{cccccc} -\Sigma\alpha & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ -\Sigma\beta & \beta_1 & \beta_2 & \ldots & \beta_n \\ -\Sigma\gamma & \gamma_1 & \gamma_2 & \ldots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \multicolumn{5}{c}{k \text{ rows}} \end{array}\right| .$$

The product of these matrices will not be affected if we prefix to the first a row of units, and to the second the row 1, 0, 0, 0, . . . ; for in the border thus introduced the row is 1, 0, 0, 0, . . ., because the sum of the constituents in any row of the second matrix is 0. Hence,

$$\frac{H}{(-u)^k} = \left|\begin{array}{cccccc} 1 & 1 & 1 & \ldots & 1 \\ \Sigma\alpha & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \Sigma\beta & \beta_1 & \beta_2 & \ldots & \beta_n \\ \Sigma\gamma & \gamma_1 & \gamma_2 & \ldots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \multicolumn{5}{c}{(k+1) \text{ rows}} \end{array}\right| \cdot \left|\begin{array}{cccccc} 1 & 0 & 0 & \ldots & 0 \\ -\Sigma\alpha & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ -\Sigma\beta & \beta_1 & \beta_2 & \ldots & \beta_n \\ -\Sigma\gamma & \gamma_1 & \gamma_2 & \ldots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \multicolumn{5}{c}{(k+1) \text{ rows}} \end{array}\right| .$$

We may add to any *row* of either of these matrices any linear combination of the remaining rows, just as with determinants (although the like is not true for columns); because it is the same to do this as to do it in each of the separate determinants the sum of whose products is equal to the product of the matrices. Multiplying, then, in the first matrix, the α row by x, the β row by y, the γ row by z, etc., and subtracting the sum of these products from the first row, we have, since

$$\alpha_i x + \beta_i y + \gamma_i z + \ldots = 1,$$

$$\frac{H}{(-u)^k} = \left|\begin{array}{cccccc} -(n-1) & 0 & 0 & \ldots & 0 \\ \Sigma\alpha & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \Sigma\beta & \beta_1 & \beta_2 & \ldots & \beta_n \\ \Sigma\gamma & \gamma_1 & \gamma_2 & \ldots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \multicolumn{5}{c}{(k+1) \text{ rows}} \end{array}\right| \cdot \left|\begin{array}{cccccc} 1 & 0 & 0 & \ldots & 0 \\ -\Sigma\alpha & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ -\Sigma\beta & \beta_1 & \beta_2 & \ldots & \beta_n \\ -\Sigma\gamma & \gamma_1 & \gamma_2 & \ldots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \multicolumn{5}{c}{(k+1) \text{ rows}} \end{array}\right| .$$

$$= -(n-1)\,\Sigma \left|\begin{array}{cccc} \alpha_1 & \alpha_2 & \ldots & \alpha_k \\ \beta_1 & \beta_2 & \ldots & \beta_k \\ \gamma_1 & \gamma_2 & \ldots & \gamma_k \\ \cdot & \cdot & \cdot & \cdot \\ \multicolumn{4}{c}{k \text{ rows}} \end{array}\right|^2 .$$

Hence,

$$H = -(n-1)(-u)^{k-2} \Sigma (J_{12\ldots k} L_{k+1} L_{k+2} \ldots L_n)^2,$$

where $J_{12\ldots k}$ is the determinant of the coefficients of the k factors $L_1, L_2, \ldots L_k$.

For example, the Hessian of

$$u = (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z)(a_4x + b_4y + c_4z)$$

is equal to

$$-3u\left[\begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix}^2 (a_1x+b_1y+c_1z)^2 + \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix}^2 (a_2x+b_2y+c_2z)^2 \right.$$
$$\left. + \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}^2 (a_3x+b_3y+c_3z)^2 + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 (a_4x+b_4y+c_4z)^2 \right].$$

From the fact that $-H/(-u)^{k-2}$ is a sum of squares, the following points are obvious in regard to a system of n *real* linear loci, or flats of $k-2$ dimensions, in space of $k-1$ dimensions, n being supposed equal to or greater than k:—

1°. The Hessian of the system vanishes identically if, and only if, all the n flats pass through one point.

2°. If no k of the flats meet in a point, the Hessian consists of the original system of flats counted $k-2$ times, together with an additional locus of the order $2(n-k)$, which is entirely imaginary.

3°. If there be points in which k or more of the flats meet, the additional locus passes through these points, but is otherwise entirely imaginary.

Of course, a particular case of 2° is the theorem, that if the roots of a real binary quantic are all real and distinct, those of its Hessian are all imaginary.